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(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FOURTH SEMESTER EXAMINATION, MAY 2012

SECOND YEAR

Date : 21/05/2012 **MATHEMATICS (Honours)**

Time : 11 am – 3 pm Paper : IV Full Marks : 100

[Use Separate Answer Books for each group]

Group - A

(Answer **any five** questions from Question No. 1 to 8 and **any three** from Question No. 9 to 13)

1. a) If D is dense in $\mathbb{R}$, prove that for any $x \in \mathbb{R}$, the set $\{x+y: y \in D\}$ is also dense in $\mathbb{R}$. Hence find two countable, disjoint, dense sets in $\mathbb{R}$. [3+2]

   b) In $\mathbb{R}^2$, find $d(A, B)$ where $A = \{(x,y): x = 0, y \in \mathbb{R}\}$ and $B = \{(x,y): x \in \{1, \frac{1}{2}, \frac{1}{3}, \dots \dots\}, y \in \mathbb{R}\}$, d is the usual metric on $\mathbb{R}^2$. [2]

2. Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $d(x,y) = \min\{1, |x-y|\}$.

   a) Show that d defines a metric on $\mathbb{R}$.

   b) What is the diameter of $\mathbb{R}$ under this metric?

   c) Is d equivalent to the usual metric on $\mathbb{R}$? Justify your answer. [3+1+3]

3. Let us consider the Euclidean metric on $\mathbb{R}^2$ and let A be a non-empty closed subset of $\mathbb{R}^2$.

   a) Prove that $x \in \mathbb{R}^2 \setminus A$ if and only if $d(x, A) > 0$.

   b) Let B be another closed subset of $\mathbb{R}^2$. Then does $A \cap B = \phi \Rightarrow d(A, B) > 0$? Justify your answer. [4+3]

4. a) Define a 2nd countable and a separable metric space. Show that every separable metric space is 2nd countable. [2+3]

   b) Let G be a dense open set in $\mathbb{R}$. Show that for each $x \in \mathbb{R}$, there are $a, b \in G$ such that $x = a - b$. [2]

5. a) Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots$ be a nested family of closed sets in a metric space (X,d) with $d(A_n) < \infty \forall n$. Where $d(A_n)$ denotes the diameter of A_n in (X,d). Then does this imply $\bigcap_{n=1}^{\infty} A_n \neq \phi$? Justify.

   b) Can the set of irrational numbers be written as $\bigcup_{n=1}^{\infty} F_n$ where F_n are closed subsets of $\mathbb{R}$? [3+4]

6. a) Let $f, g: X \to Y$ be continuous functions where X, Y are metric spaces. Prove that the set $\{x \in X: f(x) = g(x)\}$ is closed in X.

   Use above result to show that if $f, g: \mathbb{R} \to \mathbb{R}$ are continuous and if $f|_{\mathbb{Q}} = g|_{\mathbb{Q}}$ then $f = g$. [3+2]

   b) Let X be a metric space, $A \subseteq X$. Prove that the function $f: X \to \mathbb{R}$ defined by $f(x) = d(x, A)\ \forall\ x \in X$ is uniformly continuous. [2]

7. a) Define Lebesgue number of an open cover in a metric space (X, d). Prove that if (X, d) is compact then the Lebesgue number exists for any open cover.

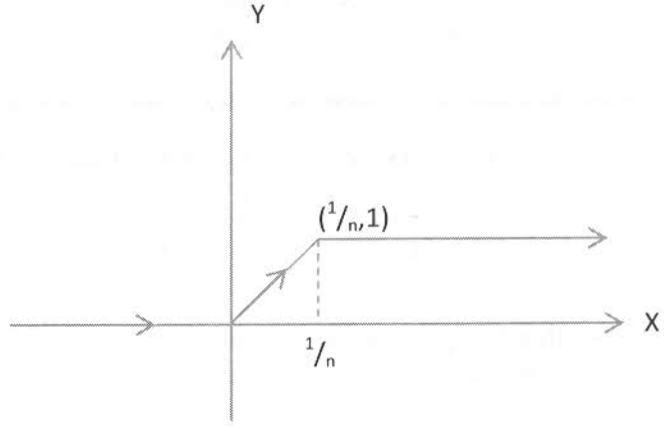
   b) By the use of the fact in (a) prove that any continuous function $f: (X,d) \to (y,d')$ is uniformly continuous if (X, d) is compact. [4+3]

8. a) Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(\mathbb{Q}) \subseteq \mathbb{R} - \mathbb{Q}$ and $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$. Prove that f can't be continuous. [2]

   b) Show that a connected metric space with at least two points is uncountable. [3]

   c) Prove that the space $\mathbb{Q}$ is not connected. [2]

9. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on $\mathbb{R}$. The graph of f_n is shown in the figure. Write the expression for $f_n(x)$. If $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f, then compute f. Does $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f? Justify your answer. [1+1+3]



10. Let $\{f_n\}_{n\in\mathbb{N}}$, where $f_n:\mathbb{R}\to\mathbb{R}$, be a sequence of differentiable functions and $\sum_{n=1}^{\infty} f_n$ converges uniformly to f. Is $\sum_{n=1}^{\infty} f_n'$ convergent? Justify. If $\sum_{n=1}^{\infty} f_n'$ convergent then does the equality $\sum_{n=1}^{\infty} f_n' = f'$ holds? [3+2]

11. a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a set $E\subset\mathbb{R}$ and let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. If $\sum_{n=1}^{\infty} M_n$ be convergent and if $|f_n(x)|\le M_n$ for each $x\in E$ and $n=1,2,\cdots$, then show that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E. [3]

    b) Prove or disprove: $\sum_{n=1}^{\infty}\frac{1}{2^n}\cos(3^n x)$ represents an everywhere continuous function. [2]

12. Answer either (a) or (b):

    a) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series such that $\overline{\lim}\,|a_n|^{\frac{1}{n}} = R,\ 0<R<\infty$. Prove that the given series is absolutely convergent for $|x|<\frac{1}{R}$ and divergent for $|x|>\frac{1}{R}$. [5]

    b) If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at the end point $x = R$ of the interval of convergence $(-R,R)$, prove that the series is uniformly convergent on [0, R]. [5]

13. If $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2} = f(x)$, then prove that $\int_0^{\pi} f(x)dx = 2\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right)$. [5]

Group – B

(Attempt **any three** from Q. No. 14 to 18 and **any two** from Q. No. 19 to 21)

14. a) Solve $x\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+2y = x^3e^x$ after the determination of a solution of its reduced equation. [5]

    b) Find the eigen values and eigen functions for the differential equation $\frac{d}{dx}(x\frac{dy}{dx})+\frac{\lambda}{x}y = 0,\ (\lambda>0)$. Satisfying the boundary conditions $y(1)=0$ and $y(e^{\pi})=0$. [5]

15. a) Solve $\frac{d^2y}{dx^2}-\frac{2}{x}\frac{dy}{dx}+(a^2+\frac{2}{x^2})y=0$ by reducing to normal form. [5]

b) Solve: $\frac{dx}{dt}+5x+y=e^t$. [5]

$\frac{dy}{dt}-x+3y=e^{2t}$

16. a) Solve the differential equation:
$(2xz-yz)dx+(2yz-zx)dy-(x^2-xy+y^2)dz=0$. [5]

b) Find the equation of the integral surface of the differential equation $2y(z-3)p+(2x-z)q=y(2x-3)$ which passes through the circle $z=0$, $x^2+y^2=2x$, where $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$. [5]

17. a) Show that the general solution of the linear partial differential equation $Pp+Qq=R$ is $F(u,v)=0$ where F is an arbitrary function and $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$ are two independent integrals of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. [5]

b) Find a complete integral of $xpq+yq^2=1$ by Charpit's method where $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$. [5]

18. a) Solve the PDE: $(y+z+u)\frac{\partial u}{\partial x}+(z+u+x)\frac{\partial u}{\partial y}+(u+x+y)\frac{\partial x}{\partial z}=x+y+z$. [4]

b) Solve: $4\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=16\log(x+2y)$. [4]

c) Find the general solution of following linear PDE:
$[D^4+D'^4]z=0$ where $D\equiv\frac{\partial}{\partial x}$, $D'\equiv\frac{\partial}{\partial y}$. [2]

19. a) Find the pedal equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to its centre as pole. [3]

b) If $x\cos\theta+y\sin\theta=p$ touches the curve $x^m y^n=a^{m+n}$, prove that $p^{m+n}m^m n^n=(m+n)^{m+n}\, a^{m+n}\sin^n\theta\cos^m\theta$. [4]

c) Find the radius of curvature of $y=xe^{-x}$ at its maximum point. [3]

20. a) Show that the points of intersection of the curve:
$2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$
and its asymptotes lie on the straight line $8x+2y+1=0$. [5]

b) If envelope of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^{2/3}+y^{2/3}=c^{2/3}$ then prove that the parameters *a*, *b* are connected by $a+b=c$. [5]

21. a) Find the values of x for which the curve $y=x^4-6x^3+12x^2+5x+7$ is concave upword or downwords. [3]

b) Find the volume of the solid obtained by the revolution of the Cissoid $y^2(2a-x)=x^3$ about its asymptote. [4]

c) Show that the area bounded by the circle $x^2+y^2=64a^2$ and the parabola $y^2=12ax$ $(a>0)$ lying in the positive side of x-axis is $\frac{16a^2}{3}(4\pi+\sqrt{3})$. [3]
